

Stringy Symmetries and the Higher Spin Square

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ABSTRACT: Tensionless string theory on $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$, as captured by a free symmetric product orbifold, has a large set of conserved currents which can be usefully organised in terms of representations of a $\mathcal{N} = (4, 4)$ supersymmetric higher spin algebra. In this paper we focus on the single particle currents which generate the asymptotic stringy symmetry algebra on AdS_3 , and whose wedge modes describe the unbroken gauge symmetries of string theory in this background. We show that this global subalgebra contains two *distinct* higher spin algebras that generate the full algebra as a ‘higher spin square’. The resulting unbroken stringy symmetry algebra is exponentially larger than the two individual higher spin algebras.

Contents

1	Introduction	1
2	Single Particle Symmetry Generators	6
3	Building Blocks	9
3.1	Free Boson	10
3.2	Free Fermion	11
3.3	Bosonisation	12
3.4	Putting the Blocks Together	13
4	The Higher Spin Square	14
4.1	Free Bosons	15
4.2	The Clifford Algebra Square – a Toy Model	16
4.3	The $\mathcal{N} = 4$ Higher Spin Square	18
5	Concluding Remarks	19

1 Introduction

The uniqueness of string (or M-)theory is one of its most attractive features, and yet the reason that it is so, is not transparent. Indeed, until the advent of string dualities it was not even clear that the handful of different looking perturbative string theories were limits of a single theory. Whenever we encounter in physics a rigidity of structure it is most often due to an underlying symmetry (or invariance) principle. Hence, we might try and look for such a symmetry principle constraining string theory. However, if a powerful constraining symmetry is present in string theory it is mostly hidden from view. Perturbative string theories expanded about flat space manifest diffeomorphism and gauge invariance in the massless sector, but little else. But there have been persistent hints [1–4], e.g., in high energy scattering, of the restoration of a larger broken symmetry. If this is so, one needs a more symmetric vacuum to expand around, to manifest (at least some more of) the additional symmetries.

String theories in AdS backgrounds might be better test cases for this purpose. Indirectly, through gauge-string duality, we are led to the conclusion that the AdS

string theory dual to a free large N QFT would have a large set of unbroken gauge symmetries [5–8]. These are the symmetries of the Vasiliev higher spin theory [9]. In fact, the Vasiliev theories of interacting higher spin gauge fields are a very elegant realisation of how invariance principles can largely constrain the form of a theory. At least when coupled to a finite number of low spin matter fields the classical equations of motion appear to be unique (perhaps up to some parameters).¹ But string theory is not simply a Vasiliev theory — it has an infinite set of matter fields (of arbitrarily high spin) coupled to the Vasiliev gauge fields. Also the gauge symmetries of the Vasiliev theory are only one infinite tower or one Regge trajectory of the string theory. So this is not quite a large enough unbroken stringy symmetry — a genuinely stringy symmetry should be exponentially larger in size. Are there vacua which exhibit such a larger unbroken symmetry?

String theories on AdS_3 have an even larger unbroken symmetry than a generic AdS_{d+1} background. Again this can be indirectly inferred from gauge-string duality. For free gauge theories in $d \geq 3$, the only single trace conserved currents are built from bilinears and their derivatives, as follows from unitarity bounds for conserved currents in a CFT. In the bulk these correspond exactly to the Vasiliev gauge symmetries and no more. But CFTs in two dimensions which are dual to AdS_3 string backgrounds are, in many cases, believed to be deformations of free symmetric product orbifolds, see [21] for a review. These have a very large set of conserved currents at their orbifold point — those built from bilinears of the underlying free bosons or fermions form only a tiny subset. As we will see below, this is true even after isolating the single particle ('single trace') generators from amongst all the currents. Assuming that this free point corresponds to a tensionless point in the moduli space of AdS_3 string vacua, we conclude that there must be a much larger unbroken gauge symmetry in the bulk string theory here than in the higher dimensional cases [22].

We should, however, keep in mind that string theory (or any theory of gravity, for that matter) on AdS_3 always exhibits an enhanced asymptotic symmetry algebra. Indeed, as first observed by Brown and Henneaux, diffeomorphism invariance on AdS_3 has its global algebra of $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ enhanced to two copies of the Virasoro algebra with a universal central charge [23]. Similarly, the higher spin gauge invariances of a Vasiliev theory have their global symmetry algebra of $\mathfrak{sl}(N) \oplus \mathfrak{sl}(N)$ (or more generally, $\text{hs}[\lambda] \oplus \text{hs}[\lambda]$) enhanced to two copies of a \mathcal{W}_N (or $\mathcal{W}_\infty[\lambda]$, respectively) asymptotic symmetry algebra [24–27]. In order to extract the global part of the asymptotic symmetry algebra of string theory we have to restrict it to its ‘wedge’ modes — these consist

¹The classical and 1-loop Vasiliev theory successfully capture the large N limit of vector-like models with no propagating gauge fields [10–18], see [19, 20] for reviews.

of the ‘single particle’ generators that annihilate both the in- and out-vacuum — and take the central charge to infinity. The resulting ‘wedge’ algebra — for example, for the case of the bosonic \mathcal{W}_N algebra this is just $\mathfrak{sl}(N)$ — will then tell us about the unbroken stringy symmetries in the bulk. It is also this algebra which is gauged and which will, hopefully, help in constraining the structure of string theory — just as the Vasiliev theory based on $\text{hs}[\lambda] \times \text{hs}[\lambda]$ was largely so, by its algebra.

In this paper we make a beginning towards getting an explicit handle on this unbroken stringy algebra for the case of superstrings on $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$ in its tensionless limit, as captured by the symmetric product orbifold SCFT of \mathbb{T}^4 . We will exploit the fact that the whole symmetric orbifold chiral algebra can be organised in terms of representations of the superconformal $\mathcal{W}_\infty^{(\mathcal{N}=4)}$ higher spin algebra [22]; the relevant representations are of the form $(0; \Lambda)$, where Λ is a representation of $\text{SU}(N)$ [22, 28]. First, we will identify which of these representations correspond to *single particle* symmetry generators. These are the analogues of ‘single trace’ operators in gauge theories, and are built from a single cycle of the symmetric group. Their wedge modes then generate the global part of the symmetry algebra which, as mentioned above, is the unbroken gauge algebra of the bulk string theory.

For the case of the symmetric orbifold of \mathbb{T}^4 , the single particle generators are in one-to-one correspondence with the elements of the chiral algebra of \mathbb{T}^4 . (Indeed, to each element ρ in the chiral algebra of \mathbb{T}^4 , we can associate a generator of the symmetric orbifold algebra via the symmetrised sum, $\sum_i \rho^i$.) Note that when we take the large N limit of the symmetric orbifold, all these generators become independent (unlike the generators of the chiral algebra of a single \mathbb{T}^4 which cannot all be independent). The results of [22] imply that these resulting generators must organise themselves in terms of representations of the wedge subalgebra $\text{shs}_2[\lambda = 0]$ of $\mathcal{W}_\infty^{(\mathcal{N}=4)}[0]$ — the representation of the full $\mathcal{W}_\infty^{(\mathcal{N}=4)}[0]$ algebra will also contain multi-particle states, and it is only the wedge modes that map single particle generators to themselves. We find that the relevant representations (which carry the same label both for $\mathcal{W}_\infty^{(\mathcal{N}=4)}$ and its wedge subalgebra), are $(0; \Lambda)$, where Λ is described in terms of Dynkin labels as $\Lambda = [m, 0, \dots, 0, n]$, and each representation appears with multiplicity one. In simpler terms, these are the completely symmetric powers of the fundamental/antifundamental representation. Here the generators from $(m = 1, n = 0)$, $(m = 0, n = 1)$ and $(m = n = 1)$ are those of the $\mathcal{W}_\infty^{(\mathcal{N}=4)}$ algebra itself (the free fermions, and the invariant bilinears together with their superconformal descendants). The others account for the additional single particle generators of the full symmetric orbifold chiral algebra.

In order to describe the Lie algebra structure of these wedge generators, it is convenient to go back to the full asymptotic symmetry algebra. Furthermore, it is instructive

to analyse first the structure of a simpler bosonic toy model, the N 'th symmetric orbifold of a single real boson. In this case the single particle generators of the chiral algebra are in one-to-one correspondence with the chiral sector of a single boson. While this provides a complete characterisation of the algebra, it is useful to refine this description in terms of its higher spin subalgebras. This is what the ‘higher spin square’ structure uncovers.

A natural \mathcal{W}_∞ subalgebra, in this case, is spanned by the bilinear expressions

$$\sum_{i=1}^N (\partial^{m_1} \phi^i) (\partial^{m_2} \phi^i) , \quad m_1, m_2 \geq 1 , \quad (1.1)$$

suitable linear combinations of which define the quasiprimary generators of spin $s = m_1 + m_2$. They generate the even spin subalgebra $\mathcal{W}_\infty^{(e)}[1]$ of $\mathcal{W}_\infty[1]$. On the other hand, the most general ‘single-particle’ generators involve the symmetrised products of order $p \geq 1$

$$\sum_{i=1}^N (\partial^{m_1} \phi^i) \cdots (\partial^{m_p} \phi^i) , \quad m_1, \dots, m_p \geq 1 . \quad (1.2)$$

The (singular part of the) OPE of a \mathcal{W}_∞ generator with an order p generator (1.2) contains, up to possible terms of lower order, only generators of order p ; corrected by suitable lower order terms, the generators of order p therefore form a representation of \mathcal{W}_∞ .² In order to visualise this, we can arrange these representations in parallel vertical columns, where each column comprises of ‘descendants’ with the $\mathcal{W}_\infty^{(e)}[1]$ algebra acting ‘vertically’, see Figure 1. In other words, the commutators with the generators from the higher spin algebra $\mathcal{W}_\infty^{(e)}[1] = \mathcal{W}_\infty^{(\text{vert})}$ move one up and down along each of the green columns. Note that each column really becomes ‘thicker’ as we move vertically downwards, corresponding to the exponentially increasing number of descendants, i.e., of splitting up the total spin among the p integers m_1, \dots, m_p . These columns generalise to the representations $(0; \Lambda)$ with $\Lambda = [m, 0, \dots, 0, n]$ in the $\mathcal{N} = 4$ case.

The crucial observation is now that the commutators involving the ‘top’ generators from each column *also* define a higher spin symmetry algebra, the ‘horizontal’ \mathcal{W} -algebra $\mathcal{W}_\infty^{(\text{hor})}$. In fact, the generators associated to $\sum_i (\partial \phi^i)^l$ close, up to lower order correction terms, to form the \mathcal{W}_∞ algebra $\mathcal{W}_{1+\infty}[0]$; this was first observed in [29], and is basically a consequence of the fact that we may equivalently describe the real boson theory in terms of a complex fermion, whose associated \mathcal{W}_∞ algebra is $\mathcal{W}_{1+\infty}[0]$. We can therefore equally well organise the full stringy algebra in terms of representations

²Strictly speaking, they only form a representation of the wedge subalgebra of \mathcal{W}_∞ . If we want to think of them as honest representations of \mathcal{W}_∞ we must also include the multiparticle states that are made up of one generator of order p , together with an arbitrary number of bilinear \mathcal{W}_∞ generators.

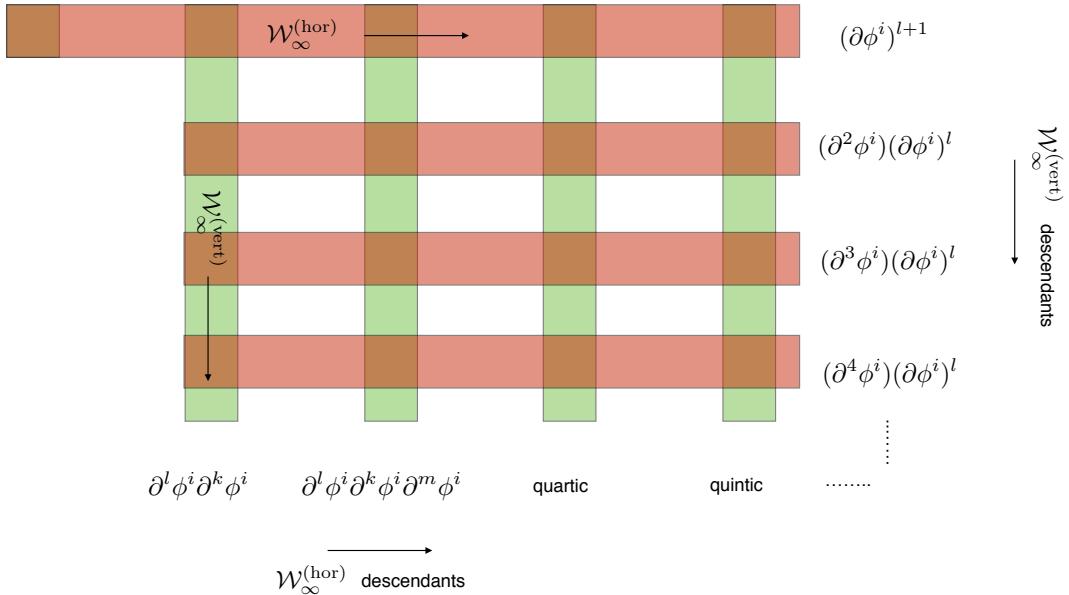


Figure 1. The Higher Spin Square.

of this horizontal \mathcal{W} -algebra (or rather again its wedge subalgebra), i.e., in terms of the red rows of Figure 1, see eq. (3.21) below. From this perspective, the left-most terms in each row (except for the first row) are the bilinear currents of the form (1.1) with spin $s \geq 3$, while the top left entry corresponds to $\sum_i \partial^l \phi^i$, i.e., to the spin $s = 1$ generator of $\mathcal{W}_{1+\infty}[0]$, together with its L_{-1} descendants — these states sit in a single irreducible representation of the (wedge subalgebra of the) vertical algebra $\mathcal{W}_\infty^{(\text{vert})}$.

The picture we thus arrive at is that the commutators along a given row are governed by the horizontal higher spin algebra $\mathcal{W}_\infty^{(\text{hor})}$, while commutators down a given column are governed by the vertical higher spin algebra $\mathcal{W}_\infty^{(\text{vert})}$. Knowing both thus determines, in principle, an arbitrary commutator between elements of the chiral algebra. We may use, for instance, $\mathcal{W}_\infty^{(\text{vert})}$ to bring any element to the topmost row of its column, and then employ $\mathcal{W}_\infty^{(\text{hor})}$ to evaluate commutators with any other entry.³ Therefore, the global horizontal and vertical higher spin symmetry algebras together effectively generate the full stringy symmetry — the wedge algebra of the above chiral

³Or we may just as well employ $\mathcal{W}_\infty^{(\text{hor})}$ to bring any element to the left most column, and then utilise $\mathcal{W}_\infty^{(\text{vert})}$.

algebra. We will thus refer to this symmetry structure as a ‘higher spin square’.⁴ We believe the existence of this higher spin square is an indication that the stringy symmetries are best understood by decomposing them in terms of the higher spin symmetries.

While the analysis for this bosonic toy model is rather natural and symmetrical (and can be checked fairly directly, see in particular Section 4.1), we have so far not succeeded in finding an equally canonical description in the $\mathcal{N} = 4$ case, although a number of possible constructions exist (see Section 4.3). On the other hand, we have found a neat finite-dimensional analogue — associated to the Clifford algebra — which builds intuition for this structure, and that is studied in some detail in Section 4.2. In that case, one has γ matrices in $2n$ dimensions. The bilinears in the γ matrices are the generators of $\mathfrak{so}(2n)$ which plays the role of the vertical algebra. The other multi-linears define various antisymmetric representations of $\mathfrak{so}(2n)$ and can be arranged in vertical columns. The γ -matrix algebra gives us a natural set of commutators for elements across different columns — a horizontal algebra — and the full Clifford algebra is generated by the horizontal and vertical algebras. In this case it is easy to see that the resulting Lie algebra — the ‘Clifford algebra square’ — is just $\mathfrak{su}(2^n)$ since the gamma matrices generate all linearly independent matrices of size 2^n . This shows that there is an exponential enhancement in rank in going from the vertical algebra $\mathfrak{so}(2n)$ to the full Clifford algebra $\mathfrak{su}(2^n)$. We find this to be a useful model to keep in mind as some kind of a finite truncation of the higher spin square.

The outline of this paper is as follows. Section 2 contains the identification of the single particle symmetry generators for the symmetric orbifold of \mathbb{T}^4 . Section 3 discusses the decomposition of the chiral algebra for free bosons and free fermions in terms of both horizontal and vertical bosonic \mathcal{W}_∞ representations. These building blocks are then put together for the SUSY $\mathcal{N} = 4$ case. Section 4 uses these results and describes the higher spin square, first in the case of the free boson (Section 4.1) and then for the $\mathcal{N} = 4$ case (Section 4.3). We also introduce the Clifford algebra square in Section 4.2. Finally Section 5 contains miscellaneous closing remarks.

2 Single Particle Symmetry Generators

Let us begin by identifying the single particle currents of the symmetric orbifold of \mathbb{T}^4 . The basic idea is to write the chiral sector of the orbifold partition function for

⁴The terminology must not mislead one into viewing this as a square (or tensor product) of the two higher spin symmetries. The rank is, in a precise sense, exponentially larger than that of either the vertical or horizontal symmetry algebra since it involves generators built effectively from multilinear products of the individual higher spin generators.

$\text{Sym}^N(\mathbb{T}^4)$ (in the limit of large N) in a second quantised form. Since the operation of taking the symmetric product is essentially one of ‘multi-particling’, it is not greatly surprising that the generating function of the single particle currents is the same as that of four free bosons and fermions — the content of the supersymmetric \mathbb{T}^4 theory. The new observation will be the decomposition of these generators into specific representations of the small $\mathcal{N} = 4$ shs₂[$\lambda = 0$] algebra, the wedge subalgebra of the supersymmetric \mathcal{W}_∞ algebra.

Let us start with the familiar formula for the generating function for the untwisted sector of a general symmetric orbifold [30]

$$\sum_{N=0}^{\infty} p^N Z_{\text{RR}}^{(\text{U})}(\text{Sym}^k(X)) = \prod_{\Delta, \bar{\Delta}, \ell, \bar{\ell}} \frac{1}{(1 - pq^\Delta \bar{q}^{\bar{\Delta}} y^\ell \bar{y}^{\bar{\ell}})^{c(\Delta, \bar{\Delta}, \ell, \bar{\ell})}} , \quad (2.1)$$

where the coefficients $c(\Delta, \bar{\Delta}, \ell, \bar{\ell})$ are the expansion coefficients of the R-R partition function of X (with the insertion of $(-1)^{F+\tilde{F}}$),

$$Z_{\text{RR}}(X) = \sum_{\Delta, \bar{\Delta}, \ell, \bar{\ell}} c(\Delta, \bar{\Delta}, \ell, \bar{\ell}) q^\Delta \bar{q}^{\bar{\Delta}} y^\ell \bar{y}^{\bar{\ell}} . \quad (2.2)$$

We are interested in the \mathcal{W} algebra of the symmetric orbifold of $X = \mathbb{T}^4$. In particular, we want to analyse only the purely left-moving states and we want to describe them in the NS-sector. Explicitly applying the spectral flow, this leads to

$$\sum_{N=0}^{\infty} p^N Z_{\text{vac,NS}}^{(N)}(\tau, y) = \prod_{n=0}^{\infty} \prod_{\ell \in \mathbb{Z}} \frac{1}{(1 + (-1)^\ell p q^{n+\ell/2+1/4} y^{\ell+1})^{c(n, \ell)}} , \quad (2.3)$$

where the $c(n, \ell)$ are the expansion coefficients of the R-sector chiral partition function of \mathbb{T}^4 (with the insertion of $(-1)^F$)

$$(y - 2 + y^{-1}) \prod_{n=1}^{\infty} \frac{(1 - yq^n)^2 (1 - y^{-1}q^n)^2}{(1 - q^n)^4} = \sum_{n=0}^{\infty} \sum_{\ell \in \mathbb{Z}} c(n, \ell) q^n y^\ell . \quad (2.4)$$

Note that the term with $n = 0$ and $\ell = -1$ in eq. (2.3) describes the contribution of the NS vacuum with $q^{-1/4}$ — it follows from eq. (2.4) that $c(0, \pm 1) = 1$ — while all other terms in eq. (2.3) lead to positive powers of q . Thus the NS vacuum character stabilises for sufficiently large N to

$$q^{N/4} Z_{\text{vac,NS}}^{(N)}(\tau, y) = \prod_{n=0}^{\infty} \prod_{\ell \in \mathbb{Z}}' \frac{1}{(1 + (-1)^\ell q^{n+\ell/2+1/2} y^{\ell+1})^{c(n, \ell)}} , \quad (2.5)$$

where the prime at the product means that now the term with $n = 0$ and $\ell = -1$ is excluded. This can now also be written directly in terms of the NS sector expansion coefficients

$$q^{N/4} Z_{\text{vac,NS}}^{(N)}(\tau, y) = \prod_{r=\frac{1}{2}}^{\infty} \prod_{\ell \in \mathbb{Z}} \frac{1}{(1 - (-1)^{2r} q^r y^\ell)^{d(r,\ell)}} , \quad (2.6)$$

where r runs over half-integers and integers and the coefficients $d(r, \ell)$ are now the expansion coefficients of the spectrally flowed version of (2.4)

$$\prod_{n=1}^{\infty} \frac{(1 - y q^{n-\frac{1}{2}})^2 (1 - y^{-1} q^{n-\frac{1}{2}})^2}{(1 - q^n)^4} = 1 + \sum_{r=\frac{1}{2}}^{\infty} \sum_{\ell \in \mathbb{Z}} d(r, \ell) q^r y^\ell . \quad (2.7)$$

The expression in eq. (2.6) is now very suggestive: it is already of ‘multi-particle’ form, where each state appearing in the NS character of \mathbb{T}^4 in eq. (2.7) defines a separate particle. Note that because of the L_{-1} descendants, each holomorphic current of spin s contributes not just as $1/(1 - q^s)$ (if bosonic) or $(1 + q^s)$ (if fermionic), but rather as

$$\prod_{n=0}^{\infty} \frac{1}{(1 - q^{s+n})} \quad (\text{bosonic}) \quad \text{or} \quad \prod_{n=0}^{\infty} (1 + q^{s+n}) \quad (\text{fermionic}) . \quad (2.8)$$

Thus the full chiral algebra of the symmetric orbifold is generated by $N(r, \ell)$ fields of spin r and $\mathfrak{u}(1)$ charge ℓ where

$$(1 - q) \left[\prod_{n=1}^{\infty} \frac{(1 + y q^{n-\frac{1}{2}})^2 (1 + y^{-1} q^{n-\frac{1}{2}})^2}{(1 - q^n)^4} - 1 \right] = \sum_{r=\frac{1}{2}}^{\infty} \sum_{\ell \in \mathbb{Z}} N(r, \ell) q^r y^\ell . \quad (2.9)$$

Equivalently, we can then write the character of the chiral algebra of the stringy extension as

$$q^{N/4} Z_{\text{vac,NS}}^{(N)}(\tau, y) = \prod_{r=\frac{1}{2}}^{\infty} \prod_{\ell \in \mathbb{Z}} \prod_{n=0}^{\infty} \frac{1}{(1 - (-1)^{2r} q^{r+n} y^\ell)^{(-1)^{2r} N(r,\ell)}} . \quad (2.10)$$

As mentioned earlier, this result for the single particle generators *per se* may not come as a big surprise since the process of taking the symmetric product is essentially one of ‘multi-particling’. Thus it is natural that the single particle generators of the \mathbb{T}^4 (or, more accurately, \mathbb{R}^4) theory are in one-to-one correspondence with the chiral generators of four free bosons and their four fermionic superpartners (as captured in the generating function (2.9)).⁵ What is important from our present point of view

⁵Note that we can only make a correspondence and not an identification since each independent generator is a fully symmetrised expression involving (derivatives of) all $4N$ bosons and fermions.

is to organise these symmetry generators in terms of the representations $(\Lambda_+; \Lambda_-)$ of the wedge algebra of the super $\mathcal{W}_\infty^{(4)}[0]$ algebra. There is, in fact, a simple way to characterise the representations which contribute to the single particle generators: they correspond to

$$(\Lambda_+; \Lambda_-) = (0; [m, 0, \dots, 0, n]) . \quad (2.11)$$

This is the natural class of representations to consider since they are the only ones which contain exactly a single cycle singlet of the symmetric group (see the discussion in appendix C.1 of [22]). This single cycle singlet is the analogue of the single trace operators when one gauges the symmetric group.

Combining this observation with the earlier generating function then leads to the prediction for an identity for the wedge characters⁶ of these super $\mathcal{W}_\infty^{(4)}[0]$ representations

$$\sum_{r, \ell} N(r, \ell) q^r y^\ell = (y + y^{-1}) \left[2q^{1/2} + 4 \sum_{r=\frac{3}{2}, \frac{5}{2}, \dots} q^r \right] + \sum_{n=1}^{\infty} (y^2 + 6 + y^{-2}) q^n \quad (2.12)$$

$$+ (1 - q) \sum_{m, n \geq 0} {}' \chi_{(0; [m, 0, \dots, 0, n])}^{(\text{wedge}) \mathcal{N}=4[0]}(q, y) . \quad (2.13)$$

Here the terms on the RHS in (2.12) describe the $\mathcal{W}_\infty^{(4)}[0]$ generators, while the contributions from the sum in (2.13) account for the wedge modes of the single-particle representations (2.11), with the terms generated from L_{-1} subtracted out.⁷ Note that we have to restrict to the ‘wedge’ part of the characters, since the other contributions correspond to ‘multi-particle’ states involving, in addition to a generator from $(0; [m, 0, \dots, 0, n])$, an arbitrary number of $\mathcal{W}_\infty^{(4)}[0]$ fields.

We have tested this identity, using eq. (2.9) as well as the wedge characters from [22], up to order q^6 . In the next section we shall also see how to understand it in terms of bosonic and fermionic building blocks.

3 Building Blocks

The decomposition of the modes of four free bosons and four free fermions into characters of the wedge algebra of $\mathcal{W}_\infty^{(4)}[0]$ is a generalisation of simpler decompositions of the chiral sector of a single free boson or fermion into characters of the wedge subalgebras of the corresponding bosonic \mathcal{W}_∞ algebras, as already sketched in the introduction. In

⁶The full characters would also include multi-particle terms, and thus only the wedge characters appear here.

⁷The prime in the sum indicates that the cases $(m, n) = (1, 0), (0, 1), (1, 1)$ are excluded. In effect, these terms precisely describe the contribution of the first line, i.e., of (2.12).

this section we first exhibit these simpler decompositions and then explain an identity involving two distinct \mathcal{W}_∞ algebras, see eq. (3.21), that is at the heart of the two dual higher spin square decompositions. Combining these simpler decompositions will also allow us to prove eq. (2.12), see the discussion in Section 3.4.

3.1 Free Boson

To understand the identity in its simplest incarnation, consider the chiral algebra generated by the symmetric orbifold of a single free real boson. The analogue of eq. (2.12), in conjunction with eq. (2.9), is in this case

$$(1-q) \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = \sum_r N(r) q^r = (1-q) \left(1 + \sum_{n=1}^{\infty} b_{([0^{n-1}, 1, 0, \dots, 0]; 0)}^{(\text{wedge})[\lambda=1]}(q) \right). \quad (3.1)$$

Here the first term is, up to the $(1-q)$ prefactor, the partition function of a single free boson while the last is the sum of the bosonic wedge characters $b_{(\Lambda; 0)}^{(\text{wedge})}(q)$ of the (even) $\mathcal{W}_\infty^{(\text{e})}[1]$ algebra of [31], see also [32]. Note that the \mathcal{W} -algebra generators themselves are counted by the wedge character corresponding to $n = 2$, see eq. (3.5) below,

$$b_{([0, 1, 0, \dots, 0]; 0)}^{(\text{wedge})[\lambda=1]}(q) = \frac{q^2}{(1-q)(1-q^2)}. \quad (3.2)$$

Indeed, taking away the $(1-q)$ factor which just accounts for derivatives, this leads to a spectrum of a single higher spin field for each even spin (which is what the theory of real bosons gives rise to).

In this case we can write down the branching functions (and therefore, wedge characters) explicitly following the results derived in [33]. The wedge characters are given by Schur polynomials (see eq. (2.8) and below of [33]), which are essentially $U(\infty)$ characters. Simplified expressions for these have been given in Appendix A of [33], see eqs. (A.13), (A.14) and (A.16). In particular,

$$\chi_{[n, 0, \dots, 0]}^{U(\infty)}(q) = \frac{q^{\frac{n}{2}}}{\prod_{k=1}^n (1-q^k)}, \quad (3.3)$$

using the fact that

$$\prod_{(ij) \in [n, 0, \dots, 0]} (1 - q^{h_{ij}}) = \prod_{k=1}^n (1 - q^k), \quad (3.4)$$

where h_{ij} are the hook lengths of the boxes in the Young tableaux. After multiplying by the factor of $q^{\frac{\lambda}{2}n}$ (see eq. (2.8) of [33]) and using the usual transposition rule between $U(\infty)$ and coset Young tableaux, we have (for $\lambda = 1$),

$$b_{([0^{n-1}, 1, 0, \dots, 0]; 0)}^{(\text{wedge})[\lambda=1]}(q) = \frac{q^n}{\prod_{k=1}^n (1 - q^k)}. \quad (3.5)$$

Thus the identity of eq. (3.1) reduces to showing

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = 1 + \sum_{n=1}^{\infty} \frac{q^n}{\prod_{k=1}^n (1-q^k)} . \quad (3.6)$$

This identity is in fact a particular case of the so-called q -binomial theorem (see for example Section 10.2 of [34]). Actually, in our case we can easily prove this identity using the combinatorics of a free scalar field which also sheds light on the nature of the associated chiral algebra generators. We organise the Fock space of the left moving excitations of a bosonic field as the union of n -particle excitations

$$\alpha_{-m_1} \alpha_{-m_2} \cdots \alpha_{-m_n} |0\rangle = \alpha_{-1-k_1} \alpha_{-1-k_1-k_2} \cdots \alpha_{-1-\sum_i k_i} |0\rangle . \quad (3.7)$$

On the right-hand-side we have written out the modes $m_i \geq 1$ in ascending order so that $k_i \geq 0$. The different contributions of such terms to the partition function is precisely given by $\frac{q^n}{\prod_{k=1}^n (1-q^k)}$. Thus summing over n gives us indeed eq. (3.6).

For the following it will also be important to consider the case of a complex boson, i.e., two real bosons. Using the fact that the wedge characters associated to boxes and anti-boxes simply multiply, this can then be written as

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} = \sum_{n,m=0}^{\infty} b_{([0^{n-1}, 1, 0, \dots, 0, 1, 0^{m-1}]; 0)}^{(\text{wedge})[\lambda=1]}(q) , \quad (3.8)$$

where we have used that the term with $m = n = 0$ is just the identity. In this case, the \mathcal{W} -algebra generators themselves can be obtained from the wedge character with $m = n = 1$,

$$b_{([1, 0, \dots, 0, 1]; 0)}^{(\text{wedge})[\lambda=1]}(q) = \frac{q^2}{(1-q)^2} , \quad (3.9)$$

accounting indeed for one field of each integer spin $s \geq 2$; this then generates $\mathcal{W}_{\infty}[1]$, without any truncation to even spin.

3.2 Free Fermion

There is also a fermionic version of the above identity, for which the relevant wedge characters are evaluated at $\lambda = 0$. It takes the form⁸

$$(1-q) \prod_{r=1}^{\infty} (1+y q^{r-1/2}) = \sum_r \tilde{N}(r, \ell) q^r y^{\ell} = (1-q) \left(1 + \sum_{n=1}^{\infty} b_{(0; [n, 0, \dots, 0])}^{(\text{wedge})[\lambda=0]}(q, y) \right) . \quad (3.10)$$

⁸ In the free fermion case, we have the equivalence of representations $(\Lambda; 0) \cong (0; \Lambda^*)$ [35]. Thus, at this stage, it is a matter of convenience to write this expression in terms of the wedge characters of $(0; \Lambda)$, rather than $(\Lambda; 0)$.

At $\lambda = 0$ the relevant wedge characters equal

$$b_{(0;[n,0,\dots,0])}^{(\text{wedge})[\lambda=0]}(q, y) = \frac{y^n q^{n^2/2}}{\prod_{k=1}^n (1 - q^k)}, \quad (3.11)$$

and again the term with $n = 2$ corresponds to the $\mathcal{W}_\infty^{(\text{e})}[0]$ generators themselves,

$$b_{(0;[2,0,\dots,0])}^{(\text{wedge})[\lambda=0]}(q) = \frac{y^2 q^2}{(1 - q)(1 - q^2)}. \quad (3.12)$$

The above identity is then a direct consequence of another special instance of a q -binomial theorem,

$$\sum_{k=0}^{\infty} \frac{y^k q^{k^2/2}}{\prod_{l=1}^k (1 - q^l)} = \prod_{r=0}^{\infty} (1 + y q^{r+1/2}). \quad (3.13)$$

Once again, we can easily prove this identity using the combinatorics of a free fermion field. As before, we organise the Fock space of the left moving excitations of a fermionic field as the union of n -particle excitations

$$\psi_{-m_1-\frac{1}{2}} \psi_{-m_2-\frac{1}{2}} \cdots \psi_{-m_n-\frac{1}{2}} |0\rangle = \psi_{-\frac{1}{2}-k_1} \psi_{-\frac{3}{2}-k_1-k_2} \cdots \psi_{-\frac{2n-1}{2}-\sum_i^n k_i} |0\rangle. \quad (3.14)$$

On the right-hand-side we have written out the modes $m_i \geq 0$ in ascending order so that $k_i \geq 0$. The different contributions of such terms to the partition function is precisely given by

$$\frac{y^n q^{\frac{n^2}{2}}}{\prod_{k=1}^n (1 - q^k)}.$$

Thus summing over n gives us indeed eq. (3.13).

3.3 Bosonisation

Actually, one can combine both fermionic and bosonic versions of these identities into dual identities for a single generating function. The basic idea is that, because of bosonisation, the neutral sector of the theory of a complex fermion is equivalent to that of a real boson. For a complex fermion, the relevant decomposition identity equals

$$\prod_{r=1}^{\infty} (1 + y q^{r-1/2}) (1 + y^{-1} q^{r-1/2}) = \sum_{n,m=0}^{\infty} b_{(0;[n,0,\dots,m])}^{(\text{wedge})[\lambda=0]}(q, y). \quad (3.15)$$

Here we have used that the conjugate of the wedge characters (3.11) take the form

$$b_{(0;[0,\dots,0,n])}^{(\text{wedge})[\lambda=0]}(q, y) = \frac{y^{-n} q^{n^2/2}}{\prod_{k=1}^n (1 - q^k)}, \quad (3.16)$$

as well as the fact that the wedge characters associated to boxes and anti-boxes multiply. Then noting that the 1 in eq. (3.10) just corresponds to $n = 0$, the identity eq. (3.15) follows directly from the square of (3.10). Note that in this language, the associated \mathcal{W}_∞ algebra now corresponds to the sector $(0; [1, 0, \dots, 0, 1])$ for which the wedge character equals

$$b_{(0;[1,0,\dots,0,1])}^{(\text{wedge})[\lambda=0]}(q) = \frac{q^1}{(1-q)^2}. \quad (3.17)$$

This then leads to one generator for each integer spin (including 1), i.e., to $\mathcal{W}_{1+\infty}[0]$. If we restrict the complex fermion theory to the neutral sector, we have the identity

$$\left. \prod_{r=1}^{\infty} (1 + yq^{r-1/2})(1 + y^{-1}q^{r-1/2}) \right|_{y^0} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)}, \quad (3.18)$$

i.e., the neutral sector of two complex fermions agrees precisely with the theory of a single real boson. Thus we can rewrite the bosonic partition function also in terms of the ‘neutral’ wedge characters of the $\lambda = 0$ theory,

$$(1 - q) \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)} = \sum_r N(r) q^r = (1 - q) \left(1 + \sum_{n=1}^{\infty} b_{(0;[n,0,\dots,0,n])}^{(\text{wedge})[\lambda=0]}(q) \right). \quad (3.19)$$

This is equivalent to the identity

$$\prod_{n=1}^{\infty} \frac{1}{(1 - q^n)} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{l=1}^n (1 - q^l)^2}, \quad (3.20)$$

which is indeed true (see Corollary 10.9.4 and below of [34]). Thus, for the case of a single real boson we have two alternative identities (see eq. (3.1))

$$\prod_{n=1}^{\infty} \frac{1}{(1 - q^n)} = \left(1 + \sum_{n=1}^{\infty} b_{([0^{n-1}, 1, 0, \dots, 0]; 0)}^{(\text{wedge})[\lambda=1]}(q) \right) = \left(1 + \sum_{n=1}^{\infty} b_{(0;[n,0,\dots,0,n])}^{(\text{wedge})[\lambda=0]}(q) \right). \quad (3.21)$$

This fact will play an important role in the following. Since fermionisation and bosonisation involves ‘non-perturbative’ field-redefinitions, it is natural to express the bosonic and fermionic expansions in terms of the representations $(\Lambda; 0)$ and $(0; \Lambda)$, respectively, see [36]. This motivates in retrospect the choice described in footnote 8.

3.4 Putting the Blocks Together

Given that we have identities for the partition functions of a complex boson and a complex fermion, see eqs. (3.8) and (3.15), we can now easily obtain the expression for

the case with $\mathcal{N} = 4$, see eq. (2.9). Indeed, taking squares of the two expressions we conclude that

$$\begin{aligned} & \prod_{n=1}^{\infty} \frac{(1+yq^{n-\frac{1}{2}})^2(1+y^{-1}q^{n-\frac{1}{2}})^2}{(1-q^n)^4} \\ &= \sum_{m_1, \dots, m_4, n_1, \dots, n_4} b_{([0^{n_1-1}, 1, 0, \dots, 0, 1, 0^{m_1-1}]; 0)}^{(\text{wedge})[\lambda=1]}(q) b_{([0^{n_2-1}, 1, 0, \dots, 0, 1, 0^{m_2-1}]; 0)}^{(\text{wedge})[\lambda=1]}(q) \\ & \quad b_{(0; [n_3, 0, \dots, 0, m_3])}^{(\text{wedge})[\lambda=0]}(q, y) b_{(0; [n_4, 0, \dots, 0, m_4])}^{(\text{wedge})[\lambda=0]}(q, y). \end{aligned} \quad (3.22)$$

Using the explicit expressions for these wedge characters, the product on the right-hand-side can be written as

$$\frac{q^{n_1+n_2+m_1+m_2+\frac{1}{2}(n_3^2+n_4^2+m_3^2+m_4^2)} y^{n_3+n_4-m_3-m_4}}{\prod_{j=1}^4 \prod_{k=1}^{n_j} (1-q^k) \prod_{k=1}^{m_j} (1-q^k)}. \quad (3.23)$$

Eq. (3.22) now reproduces (2.13) provided we identify

$$\chi_{(0; [m, 0, \dots, 0, 0])}^{(\text{wedge})[\mathcal{N}=4]}(q, y) = \sum_{n_1+m_1+n_3+m_3=m} b_{([0^{n_1-1}, 1, 0, \dots, 0, 1, 0^{m_1-1}]; 0)}^{(\text{wedge})[\lambda=1]}(q) b_{(0; [n_3, 0, \dots, 0, m_3])}^{(\text{wedge})[\lambda=0]}(q, y), \quad (3.24)$$

and similarly for $\chi_{(0; [0, 0, \dots, 0, n])}^{(\text{wedge})[\mathcal{N}=4]}(q, y)$, as well as their products. It follows from straightforward combinatorical considerations⁹ that eq. (3.24) is indeed the wedge character of the $\mathcal{N} = 4$ coset representation $(0; [m, 0, \dots, 0, 0])$. This therefore effectively proves eq. (2.13).

4 The Higher Spin Square

In the previous section we have seen how the single particle generators of the stringy asymptotic symmetry algebra can be organised in terms of representations of the wedge subalgebra $\text{shs}_2[0]$ of the $\mathcal{N} = 4$ superconformal $\mathcal{W}_{\infty}^{(\mathcal{N}=4)}[0]$ algebra, see eq. (3.22) and below. These single particle generators should now define a Lie algebra, the wedge algebra of the full stringy \mathcal{W} algebra (i.e., the \mathcal{W} algebra of the symmetric orbifold). This Lie algebra is the stringy generalisation of the $\text{shs}_2[0]$ algebra that underlies the construction of the Vasiliev higher spin theory, and it should be thought of as describing the symmetries of string theory on this background.

The aim of this section is to make first steps towards characterising this global Lie algebra. Since we are viewing this question from the point of view of the dual CFT, the discussion will be mainly couched in terms of the full asymptotic symmetry

⁹We thank Constantin Candu for analysing this in some detail.

algebra (whose wedge algebra we are seeking to describe). We shall be able to give a fairly compelling and natural description of the full algebra in terms of a ‘higher spin square’ for the the simpler symmetric orbifold of a single free boson, using the fact that we can equivalently describe the boson theory in terms of a complex fermion. We have also found a neat finite-dimensional toy model in terms of the familiar Clifford algebra of gamma matrices, which exhibits nicely the exponential enhancement in rank. However, while it is easy to show that a similar structure also exists in the $\mathcal{N} = 4$ case, the constructions of the horizontal \mathcal{W}_∞ algebra we have found so far are not particularly canonical.

4.1 Free Bosons

Let us first begin with the simple case of the symmetric orbifold of a single free real boson, i.e., the theory of N real bosons ϕ^j orbifolded by the permutation action S_N on the colour index j . The symmetric orbifold contains, in particular, the $\text{SO}(N)$ invariant bilinears of the free boson generators, which are known to generate $\mathcal{W}_\infty^{(\text{e})}[1]$. Furthermore, as we saw in eq. (3.6), the remaining generators can be organised in terms of the representations $([0^{n-1}, 1, 0, \dots, 0]; 0)$ of $\mathcal{W}_\infty^{(\text{e})}[1]$, with $n = 2$ corresponding to $\mathcal{W}_\infty^{(\text{e})}[1]$ itself. Note that it follows from eq. (3.7) that we can think of the generators coming from $([0^{n-1}, 1, 0, \dots, 0]; 0)$ as describing the symmetric invariants of n powers of the field $\partial\phi^j$ with additional derivatives sprinkled on them (corresponding to the k_i in eq. (3.7)).

In order to understand the resulting structure we now want to think of the generators of the full stringy \mathcal{W} algebra as arranged in an infinite table or square as follows (see Figure 1). We start with a horizontal row corresponding to the modes of $\sum_j (\partial\phi^j)^n$ with $n = 1, 2, \dots$, labelling the columns, i.e., the representations $([0^{n-1}, 1, 0, \dots, 0]; 0)$. The vertical column below each entry in this row consists of all the $\mathcal{W}_\infty^{(\text{e})}[1]$ descendants of this state, i.e., of all the states involving n bosons with an arbitrary number of derivatives sprinkled on them. As mentioned above, the $n = 2$ column corresponds to the $\mathcal{W}_\infty^{(\text{e})}[1]$ generators themselves. We will refer to this as the vertical higher spin or $\mathcal{W}_\infty^{(\text{vert})}$ symmetry.

What is intriguing about this description is that the top horizontal row of this square *also* generates a (distinct) \mathcal{W}_∞ higher spin algebra — which we will call the horizontal higher spin algebra $\mathcal{W}_\infty^{(\text{hor})}$. In fact, the $\sum_j (\partial\phi^j)^n$ fields (corrected by terms involving a smaller number of powers of $(\partial\phi^j)$ and additional derivatives) are what one obtains by bosonising the fermionic bilinears (built from a complex fermion and its conjugate) that generate a $\mathcal{W}_{1+\infty}$ algebra [29]. From this perspective, the $\sum_j (\partial\phi^j)^n$

terms correspond to the bilinear generators of the form

$$\sum_l \bar{\psi}^l \partial^{n-1} \psi^l , \quad (4.1)$$

or rather a sum of such bilinear terms with the $n - 1$ derivatives distributed between ψ^l and $\bar{\psi}^l$. The stress tensor corresponds to the $n = 2$ term which is common to the vertical $\mathcal{W}_\infty^{(e)}[1]$ algebra but the terms with $n \neq 2$ are not part of the vertical algebra. Note that the correction terms mentioned above are actually necessary for the operators to be quasi-primary with respect to the stress tensor; for example, for the first few spins one finds explicitly [29]

$$\begin{aligned} S^{(1)} &= \sum_j \partial \phi^j , & S^{(2)} &= \frac{1}{2} \sum_j (\partial \phi^j)^2 , \\ S^{(3)} &= \frac{1}{3} \sum_j (\partial \phi^j)^3 , & S^{(4)} &= \frac{1}{4} \sum_j (\partial \phi^j)^4 - \frac{3}{20} \sum_j (\partial^2 \phi^j)^2 + \frac{1}{10} \sum_j (\partial \phi^j)(\partial^3 \phi^j) . \end{aligned} \quad (4.2)$$

Thus we conclude that the stringy \mathcal{W} -algebra contains actually *two* higher spin algebras: the usual bilinear $\mathcal{W}_\infty^{(e)}[1]$ algebra — what we called the vertical algebra above — and the $\mathcal{W}_{1+\infty}$ algebra — what we called the horizontal algebra above — generated by the higher powers of $(\partial \phi)^n$, that contains in particular $\mathcal{W}_\infty[0]$, see, e.g., [37]. The two algebras do not commute with one another, but rather generate the full algebra upon taking successive commutators, i.e., what we have called the ‘higher spin square’. This structure is somewhat reminiscent of a Yangian symmetry that is believed to be present in $\mathcal{N} = 4$ super Yang-Mills theory at the planar level [38].

In principle, this formulation characterises all commutators of the stringy algebra recursively, and one can also restrict this description to the wedge algebra (where instead of $\mathcal{W}_\infty^{(e)}[1]$ and $\mathcal{W}_\infty[0]$ we deal with the wedge subalgebras $hs^{(e)}[1]$ and $hs[0]$, respectively). However, unfortunately, we have not yet managed to find any closed form expressions for these commutators. It would be very interesting to understand the structure of this algebra in more detail.

We should emphasise that the above analysis is only correct in the large N limit; for finite N the relevant \mathcal{W} algebras are truncated to finitely generated algebras (as follows, for example, from the fact that their central charge is finite). However, unfortunately, these truncations are rather complicated and in general not explicitly known.

4.2 The Clifford Algebra Square – a Toy Model

In order to get a feeling for how this construction works explicitly, it may be useful to consider a finite toy model that also exhibits the same kind of structure. Consider the

gamma matrix algebra in an even number of (euclidean) dimensions $2n$,

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}, \quad (i, j = 1, \dots, 2n). \quad (4.3)$$

Then, as is familiar, the terms with k distinct gamma matrices ($1 \leq k \leq 2n$) correspond to the k^{th} antisymmetric representation of $\mathfrak{so}(2n)$ and are of the form $\gamma^{[i_1}\gamma^{i_2}\dots\gamma^{i_k]}$. Let us denote this subset of the full Clifford algebra as $H^{(k)}$, with $H^{(0)}$ denoting the identity. Since each $H^{(k)}$ is a representation of $\mathfrak{so}(2n)$, there is a highest weight state and a ‘vertical’ column of $\mathfrak{so}(2n)$ descendants.

In the context of the Clifford construction, there is a natural commutator one can define on the level of the generators in $H^{(k)}$. This can either be done abstractly via the so-called Clifford commutators, see, e.g., [39] for an introduction, or more explicitly since we can think of the γ -matrices as matrices of dimension 2^n . On the level of vector spaces, the resulting commutators have the structure

$$[H^{(k)}, H^{(m)}] = \sum_{r=\max(0, (\frac{k+m}{2}-n))}^{\min(k,m)} (1 - (-1)^{r+km}) H^{(k+m-2r)}. \quad (4.4)$$

This therefore gives a very concrete realisation of the commutation relations of these additional horizontal generators (as well as their associated columns). Note that the term with $k + m - 2r = 0$ never arises since in that case $r = m = k$, and $r + km$ is necessarily even.

The γ -matrices can be thought of as matrices in dimension 2^n , and since

$$\dim\left(\bigoplus_{k=0}^{2n} H^{(k)}\right) = 2^{2n}, \quad (4.5)$$

we deduce that the elements of the whole Clifford algebra generate the full space of these matrices. Excluding the term with $k = 0$ that never arises in commutators, we conclude that the full Lie algebra is $\mathfrak{su}(2^n)$.

We should mention that the resulting structure is again very similar to what we saw in the previous section. In particular, the rank of this Lie algebra grows exponentially with the number of species n , whereas the rank of the vertical Lie algebra $\mathfrak{so}(2n)$ is linear in n . Again, this is nicely parallel to what one should expect for the relation between the higher spin symmetry and the stringy symmetry. The number of higher spin symmetry generators have a linear growth in the number of free fields, e.g., the theory based on N free fermions leads to \mathcal{W}_{1+N} with $N + 1$ generating fields, whereas the stringy symmetry generators have an exponential (Cardy) growth as one increases N in the symmetric product orbifold by S_N . While we have described the horizontal

action of commutators between the various columns in eq. (4.4), we have not quite identified a small horizontal algebra (with rank of order n) analogous to $\mathcal{W}_\infty^{(\text{hor})}$. It would perhaps be interesting to do so.

4.3 The $\mathcal{N} = 4$ Higher Spin Square

For the case with $\mathcal{N} = 4$ we should expect, similar to the bosonic case, to be able to construct a ‘horizontal’ \mathcal{W}_∞ algebra that will generate, together with the vertical $\mathcal{W}_\infty^{(4)}[0]$ algebra, the full set of generators (2.12) of the stringy extension. In particular, the horizontal algebra should contain at least one generator from each of the representations $(0; [m, 0, \dots, 0, n])$ with $m, n \geq 0$.

One possible choice for such a horizontal algebra is to consider the algebra that is generated by the various powers of $(\partial\phi^1)$ and $(\partial\bar{\phi}^1)$, where $(\phi^1, \bar{\phi}^1)$ is one arbitrarily chosen complex boson. The generator $(\partial\phi^1)^m (\partial\bar{\phi}^1)^n$ will then contribute to the representation $(0; [m, 0, \dots, 0, n])$, and hence this construction will account for all the generators. Furthermore, up to lower order correction terms, these generators define a bosonic \mathcal{W}_∞ algebra, as follows from the same arguments as in Section 4.1. It is also clear from the analysis of Section 3.4 that the full generating function can be expanded in terms of the corresponding characters, using twice the identity eq. (3.21).

However, there are a few rather unnatural aspects about this construction. First of all, the horizontal algebra is not really of higher spin type since the number of generators of spin s grows linearly with s — the relevant generators are schematically of the form

$$(\partial\phi^1)^m (\partial\bar{\phi}^1)^{s-m}, \quad m = 0, \dots, s. \quad (4.6)$$

One way to avoid this, is to make the vertical algebra slightly larger by including all bilinear terms of the underlying free fields, i.e., by adding in the representations $(0; [2, 0, \dots, 0])$ and $(0; [0, \dots, 0, 2])$. (It is fairly obvious that this will also lead to a consistent \mathcal{W}_∞ algebra whose number of generators does not grow with the spin.) Then we can make the horizontal algebra slightly smaller by only including the powers of a single real boson $(\partial\Phi^1)^m$. The resulting horizontal higher spin algebra is then the same as the one analysed in Section 4.1, i.e., it defines the bosonic algebra $\mathcal{W}_\infty^{(\text{e})}[1]$. In particular, it is therefore of higher spin type.

However, both of these constructions are somewhat unnatural since the resulting horizontal algebras are purely bosonic, and in particular, do not contain the $\mathcal{N} = 4$ superconformal algebra. It is possible to overcome this problem, but the only way we have found so far, is again rather ad hoc. Instead of the above algebras we consider the \mathcal{W}_∞ algebra that is generated by bilinears of the generators

$$(\partial\Phi^1)^{m_1}, \quad (\partial^{m_i}\Phi^i), \quad i = 2, 3, 4, \quad (\partial^{r_j}\psi^j), \quad j = 1, 2, 3, 4, \quad (4.7)$$

where the Φ^j , with $j = 1, 2, 3, 4$, denote the 4 real bosonic fields, and the ψ^j are the corresponding fermionic partners. The resulting algebra contains the $\mathcal{N} = 4$ superconformal algebra, and its structure is fairly similar to that of the bilinear extension of $\mathcal{W}_\infty^{(4)}[0]$ — in particular, the number of generators does not grow with spin, and the algebra is of ‘higher spin’ type. Again, it is clear that the full generating function can be expanded in terms of the corresponding characters, using as before the identity eq. (3.21). However, the construction singles out one of the four real boson fields, which is not particularly natural. So while these considerations demonstrate that also in the $\mathcal{N} = 4$ case, the higher spin square can be generated by two higher spin algebras, we suspect that there exists a more natural (and symmetrical) construction.

5 Concluding Remarks

In this paper we have analysed the symmetry algebra of string theory on $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$ in the tensionless limit. We have found that the unbroken symmetries in this AdS_3 vacuum exhibit the structure of a higher spin square which we have described. The lesson seems to be (at least as far as these symmetries are concerned) that knowing the commutators of the horizontal and vertical higher spin algebras completely determine all commutators of the exponentially larger unbroken stringy symmetry algebra. This may indicate that higher spin symmetries play an important role in string theory, beyond just describing the dynamics of the leading Regge trajectory.

Of course, one would hope to be able to exploit the power of this symmetry to deduce more about the broken symmetry or higgsed phase. For this it will be important to go beyond the in-principle determination of commutators mentioned above to more explicit characterisations of the algebra. It would also be interesting to understand the relation of this symmetry algebra to the Yangian that is believed to control the planar limit of $\mathcal{N} = 4$ super Yang Mills, as well as to the spin chain analysis of [40, 41], see also [42] for a recent review. It would be good to see whether the multi-particle algebras, studied recently by Vasiliev [43] as a way to generalise higher spin symmetry in string theory, have a connection with the higher spin square structure. A related question is to understand the representations of the higher spin square, especially those relevant for the description of the fields of string theory. In [22] we described how the twisted and untwisted sectors of the symmetric orbifold theory can be organised in terms of representations of what we would now call the vertical \mathcal{W}_∞ algebra. Once again, after considering the single particle excitations, these must assemble into bigger representations of the stringy symmetry above. It would be interesting to see how many such representations appear. One should also be able to use this information to constrain matrix elements of the perturbation by the $\mathfrak{su}(2)_R$ singlet twist two operator a

la Wigner-Eckart. A somewhat different approach to studying such perturbations might be to see whether current algebra methods, so successful for vector like theories [44], can be adapted to the case with a larger stringy symmetry and give useful constraints.

Another direction, which we have only touched on, concerns the finite truncation of the higher spin square. As mentioned, the \mathcal{W} -algebras at finite N will be finitely generated and there are relations between hitherto (at $N = \infty$) independent generators. This truncation is not entirely straightforward even for a symmetric orbifold of N free bosons. Such a deformed algebra reflects a quantum modification of the symmetry algebra and is presumably responsible for non-perturbative effects like the stringy exclusion principle. It would be interesting to see if the Clifford algebra square arises naturally in any such truncation. Another modification comes into play when we consider the theory on a finite \mathbb{T}^4 . We have been implicitly considering the theory on \mathbb{R}^4 by neglecting modes which carry charge on the \mathbb{T}^4 — as usually done in the D1-D5 system. Putting the theory on \mathbb{T}^4 again breaks part of the symmetry, and one may study how the corresponding perturbation is constrained by the unbroken symmetry.

The unbroken symmetries described here are those of a particular background of string theory. But the underlying symmetries of the theory are independent of the choice of background — the background enters only in manifesting which symmetries are unbroken. One may ask whether the higher spin square described here is the largest possible unbroken symmetry. As mentioned in the introduction, in higher dimensional AdS vacua, the maximal unbroken symmetries are smaller in size (though not subalgebras of the above stringy algebra). Are there backgrounds which manifest even larger unbroken symmetries? Even amongst AdS₃ vacua there are others (such as the case where \mathbb{T}^4 is replaced by K3) which will have different unbroken stringy symmetries at, say, the orbifold point. Finally, necessary and sufficient conditions for 2d permutation orbifold CFTs to have a holographic dual have recently been formulated in [45, 46] (using results of [47]). It would be nice to extend the above results for the symmetric orbifold to the more general permutation orbifolds with potential holographic duals.

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